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The Riemann Problem for the Planar Motion of an Elastic String

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1. INTRODUCTION

Consider a function r: $[-1, 1] \times [0, \infty) \to \mathbb{R}^2$ giving the location $r(x, t)$ at time t of a cross section x in an elastic string. In the absence of external forces, r satisfies the law of conservation of momentum

$$r_{tt} = \left[T(|r_x|) \frac{r_x}{|r_x|} \right]_x, \tag{1.1}$$

where $T = T(|r_x|)$ is the tension in the string, taken here to be a given smooth monotonically increasing function of the stretch $|r_x|$ alone. In (1.1), we have also taken the density of the material of the string to be constant. The derivation of Eq. (1.1) (and of more general equations of motion for the string) is explained in [1]. In a previous paper [5], the Riemann problem was solved for system (1.1) under two assumptions: that the graph of T has exactly one inflection point, at ξ_I, with

$$\operatorname{sgn} T''(\xi) = \operatorname{sgn} (\xi - \xi_I), \tag{1.2}$$

and that the characteristic values $\pm [T(|r_x|)/|r_x|]^{1/2}$, $\pm [T'(|r_x|)]^{1/2}$ have a fixed order, which amounts to

$$T'(\xi) > T(\xi)/\xi > 0 \tag{1.3}$$

for all $\xi > 1$. In (1.3), the equilibrium configuration with no tension is taken to be $r(x, t) = (x, 0)$, so that $T(1) = 0$.

The solution of the Riemann problem resolves an initial step discontinuity in the slope, tension and velocity of the string into a combination of centered waves. Specifically, we consider the Cauchy problem for system (1.1), with initial data of the form

$$(r_x, r_t)(x, 0) = \begin{cases} W_L & \text{if} \quad x < 0 \\ W_R & \text{if} \quad x > 0, \end{cases} \tag{1.4}$$

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Here, W_L, W_R are given in $\mathbb{R}^4$, and *only initial data with positive tension* will be considered:

$$|r_x(x, 0)| > 1 \qquad \text{for all } x. \tag{1.5}$$

Condition (1.3) ensures that system (1.1) is strictly hyperbolic, but appears to be unduly restictive according to experimental evidence [2]. To incorporate those stress-strain curves drawn in [2] that violate (1.3), we suppose here that (1.3) fails in an interval. To summarize, the following properties of T: $(0, \infty) \to \mathbb{R}$ will be assumed:

(1) T is of class C^3.

(2) $T'(\xi) > 0$ for all ξ.

(3) $T(1) = 0$.

(4) There exists $\xi_I > 1$ such that (1.2) holds for all $\xi > 1$.

(5) There are exactly two solutions $\xi_m < \xi_M$ of the equation

$$T(\xi)/\xi = T'(\xi).$$

The graph of T is sketched in Figure 1, from which we observe

$$\begin{aligned} &1 < \xi_m < \xi_I < \xi_M, \\ T'(\xi) < T(\xi)/\xi \qquad &\text{if and only if } \xi \in (\xi_m, \xi_M). \end{aligned} \tag{1.6}$$

The object of the paper is to provide a constructive proof of the following result.

THEOREM 1. *Let T satisfy assumptions* (1)–(5). *For any given W_L, W_R in $\mathbb{R}^4$ satisfying* (1.5), *the initial value problem* (1.1), (1.4) *has a unique weak solution among those functions (r_x, r_t) depending piecewise smoothly on x/t and whose jump discontinuities satisfy the natural entropy admissibility condition* (*specified in* Sect. 2). *Moreover, the tension T remains positive for $t > 0$, apart from a degenerate case in which T remains merely nonnegative.*

The proof of Theorem 1 consists of constructing the solution as a combination of centered shocks, rarefactions, and contact discontinuities separating intervals of x/t in which r_x and r_t are constant (see Fig. 1). Since Eq. (1.1) corresponds to a 4×4 first order system (2.1), this seems at first to involve a rather difficult analysis of wave curves in $\mathbb{R}^4$. However, the symmetry between left and right essentially reduces the problem to an analysis in $\mathbb{R}^2$. Specifically, two parameters $\xi = |r_x|$ and $\theta = \arg r_x$ serve to parametrize all elementary waves. Changes in ξ correspond to *longitudinal waves*, in which θ is constant; changes in θ with ξ constant are called *transverse waves.*

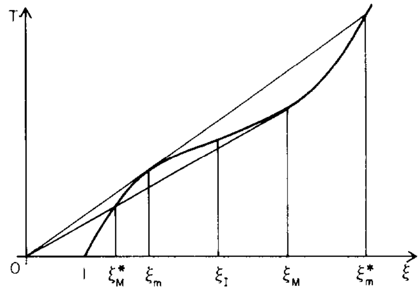

FIG. 1. The graph of T satisfying (1)–(5).

The construction of combinations of longitudinal waves given in Section 2 is essentially that of Wendroff [6] for 2×2 systems that fail to be genuinely nonlinear. For the string, genuine nonlinearity of the family of longitudinal waves breaks down at points ξ where $T''(\xi) = 0$. Any finite number of such inflection points may be accomodated in the construction of longitudinal waves by suitably adjusting the definition of $A(\xi, \xi_0)$ in Section 2. For simplicity, only a single inflection point is considered here, and this seems to be most consistent with experimental stress-strain laws [2]. The construction of combinations of longitudinal and transverse waves near points ξ where $T(\xi)/\xi = T'(\xi)$ was introduced by Keyfitz and Kranzer [4]. It is remarkable that their construction gives a unique solution of the Riemann problem no matter what the initial data, subject only to (1.5).

Note that the tension remains positive in the solution. It is straightforward to show (as was done in the strictly hyperbolic case [5]) that if the initial jump in velocity is sufficiently small, the maximum tension in the solution does not exceed that of the initial data.

Shocks, rarefaction waves and contact discontinuities are characterized in Section 2. In Section 3, the proof of Theorem 1 is outlined. This involves specifying a function $g(\xi; W_L, W_R)$ such that admissible solutions of (1.1), (1.4) correspond to solutions of

$$g(\xi; W_L, W_R) = 0. \tag{1.7}$$

It then remains only to observe that (1)–(5) imply that

(i) $g(1; W_L, W_R) \geqslant 0$

(ii) $\lim_{\xi \to \infty} g(\xi; W_L, W_R) < 0$, and

(iii) $(\partial g/\partial \xi)(\xi; W_L, W_R) < 0$ for all $\xi > 1$.

The details of the proof of (iii) are only sketched, as the specification of g is explicit enough that (iii) can be verified by a direct and straightforward case-by-case calculation. Properties (i)–(iii) mean that Eq. (1.7) is easy to solve numerically, and the construction given here of solutions of the Riemann problem is perfectly suited for an implementation of Glimm's scheme [3]. A similar construction of solutions of Riemann problems may be used to incorporate boundary conditions, so that Glimm's scheme may be used for the numerical solution of initial boundary value problems.

To describe the function g, it is convenient to introduce some notation. Write $(r_x, r_t) = (\xi\, e^{i\theta}, V) = W \in \mathbb{C} \times \mathbb{R}^2$. Subscripts on ξ, θ, V will generally correspond to subscripts on W, so that the standing assumption (1.5), for example, now reads

$$\xi_L > 1,\ \xi_R > 1. \tag{1.8}$$

For each $\xi > 1$, define $\hat{\xi}$, ξ^* by

$$T'(\xi) = [T(\xi) - T(\xi^*)]/(\xi - \xi^*)$$
$$T'(\hat{\xi}) = [T(\xi) - T(\hat{\xi})]/(\xi - \hat{\xi})$$

with $\xi^* = \infty$ or $\hat{\xi} = \infty$ if the identities cannot be satisfied. For $\xi_m \leqslant \xi \leqslant \xi_M$, define $\xi^m \leqslant \xi_m$ and $\xi^M \geqslant \xi_M$ by

$$\frac{T(\xi)}{\xi} = \frac{T(\xi^m)}{\xi^m} = \frac{T(\xi^M)}{\xi^M}. \tag{1.9}$$

Note that ξ^m and ξ^M are monotonically decreasing in $\xi \in [\xi_m, \xi_M]$ and $\xi_m^* = \xi_m^M$, $\xi_M^* = \xi_M^m$. The possibility that $\xi_m^* = \infty$ is not excluded, but since this only leads to obvious simplifications in the analysis, we mostly treat ξ_m^* as though it were necessarily finite. As a further remark, it should be noted that the absence of an inflection point in the graph of T would also merely simplify the analysis. The inflection point is included here mainly because, as remarked earlier, it appears to be a necessary physical feature, but also to show that it does not excessively complicate the mathematics. The central issue in the constructive proof of Theorem 1 is how to overcome the loss of strict hyperbolicity and consequent reordering of wave speeds due to the failure of (1.3) in the interval $[\xi_m, \xi_M]$.

2. Elementary Waves

Equations (1.1) may be written as a 4×4 first order system by setting $U = r_x$, $V = r_t$:

$$\begin{aligned} U_t &= V_x \\ V_t &= F(U)_x, \qquad F(U) = U\, T(|U|)/|U|. \end{aligned} \tag{2.1}$$

The characteristic speeds $\lambda_\pm = \pm[T'(|U|)]^{1/2}$ and $\mu_\pm = \pm[T(|U|)/|U|)]^{1/2}$ correspond to longitudinal waves and transverse waves, respectively. The structure of these waves is discussed in [2, 4, 5]. In this section, the elementary waves are characterized in a way that simplifies the method of solution of the Riemann problem.

The following classification of elementary waves distinguishes between (a) shocks, (b) centered rarefaction waves, and (c) contact discontinuities. The shocks and rarefaction waves are longitudinal. Since they lose genuine nonlinearity at the inflection point $\xi = \xi_I$ of T, the construction of rarefaction-shocks introduced by Wendroff [6] is used here. Let W_0 denote the value of $W = (U, V)$ *ahead* of a wave, and let W_1 be the value of W *behind* the wave. Subscripts 0, 1 on ξ, θ, V will refer to W_0, W_1.

(a) W_0 is joined to W_1 by a *longitudinal shock wave*, characterized by a jump in ξ across $x = st$, if

$$\begin{aligned} &\theta_1 = \theta_0,\ s = \pm[(T(\xi_1) - T(\xi_0))/(\xi_1 - \xi_0)]^{1/2} \\ &V_1 = V_0 \mp A(\xi_1, \xi_0)\, e^{i\theta_0} \qquad \text{(resp.)}, \end{aligned} \tag{2.2}$$

where

$$A(\xi_1, \xi_0) = [(T(\xi_1) - T(\xi_0))(\xi_1 - \xi_0)]^{1/2} \cdot \operatorname{sgn}(\xi_1 - \xi_0). \tag{2.3}$$

Additionally, the following *entropy admissibility condition* is imposed on all longitudinal shock waves.

$$(T(\xi) - T(\xi_0))/(\xi - \xi_0) \leqslant (T(\xi_1) - T(\xi_0))/(\xi_1 - \xi_0) \tag{2.4}$$

for all ξ between ξ_0 and ξ_1. Equivalently, ξ_1 must lie outside the interval with end points ξ_0, ξ_0^*. Condition (2.4) is natural in the sense that it is the same as the obvious "viscosity" criterion in which T is modified by the strain rate ξ_t: $\hat{T}(\xi, \xi_t) = T(\xi) + \varepsilon\xi_t$, $\varepsilon > 0$. (2.4) is also related to the "entropy condition" (see [4])

$$\eta_t + Q_x \leqslant 0$$

involving the energy $\eta = |V|^2/2 + \int^{|U|} T(\xi)\, d\xi$, and $Q = -U \cdot V\, T(|U|)/|U|$.

(b) W_0 is joined to W_1 by a *longitudinal centered rarefaction wave* if

$$\theta_1 = \theta_0 \qquad \text{and} \qquad V_1 = V_0 \pm A(\xi_1, \xi_0)\, e^{i\theta_0}, \tag{2.5}$$

where

$$A(\xi_1, \xi_0) = \int_{\xi_0}^{\xi_1} [T'(\nu)]^{1/2}\, d\nu. \tag{2.6}$$

The corresponding solution $W(x, t)$ of (2.1) is constant on each characteristic $x = \lambda_\pm(v)\, t$ with v between ξ_0 and ξ_1. Therefore $|\lambda_\pm(v)| = [T'(v)]^{1/2}$ must decrease as v goes from ξ_0 to ξ_1. That is,

$$\xi_l \geqslant \xi_1 \geqslant \xi_0 \qquad \text{or} \qquad \xi_l \leqslant \xi_1 \leqslant \xi_0 \tag{2.7}$$

for a rarefaction wave.

Next, we describe the locus of points W_1 to which W_0 may be joined by a combination of centered longitudinal waves traveling in one direction. These are the *rarefaction-shocks* involving ξ_1 between ξ_l and ξ_0^*. W_1 is given by (2.5) again, with

$$\begin{aligned} A(\xi_1, \xi_0) &= A(\hat{\xi}_1, \xi_0) + A(\xi_1, \hat{\xi}_1) \\ &= \int_{\xi_0}^{\hat{\xi}_1} [T'(v)]^{1/2}\, dv + (\xi_1 - \hat{\xi}_1)[T'(\hat{\xi}_1)]^{1/2}. \end{aligned} \tag{2.8}$$

In (2.8), we have used the fact that $\hat{\xi}_1$ lies between ξ_0 and ξ_l, so that W_0 is joined to the corresponding $\hat{W}_1$ by a rarefaction, while $\xi_1 = \hat{\xi}_1^*$ indicates that the trailing edge of the rarefaction wave is a shock, joining $\hat{W}_1$ to W_1.

Now all combinations of centered longitudinal waves traveling in one direction are described by (2.5) with $A(\xi_1, \xi_0)$ defined for all $\xi_k \geqslant 1$ $(k = 0, 1)$ by (2.3), (2.6), (2.8).

The locus of points V_1 given by (2.5), with W_0 fixed, is denoted by $\gamma_\pm$:

$$\gamma_\pm = \{ V_1 = V_1(\xi_1) \text{ given by (2.5)}, \ 1 \leqslant \xi_1 < \infty \}.$$

$\gamma_\pm$ is a straight line in the velocity plane extending from $V_1(1)$ out to infinity.

(c) W_0 is joined to W_1 by a *contact discontinuity* (or *linear* or *transverse wave*), involving a jump in θ along the line $x = st$ if

$$\xi_1 = \xi_0, \ V_1 = V_0 \pm B(\xi_0)[e^{i\theta_0} - e^{i\theta_1}] \tag{2.9}$$

where

$$s = \pm [T(\xi_0)/\xi_0]^{1/2} \qquad \text{and} \qquad B(\xi) = [\xi\, T(\xi)]^{1/2}. \tag{2.10}$$

That is, V_1 and V_0 should lie on a circle center $V_0 \pm B(\xi_0)\, e^{i\theta_0}$ with radius $B(\xi_0)$. Let $C_\pm(W_0)$ denote this circle of V_1's, parameterized by θ_1: $C_\pm = C_\pm(W_0) = \{ V_1 = V_1(\theta)\colon 0 \leqslant \theta < 2\pi \text{ given by (2.9)}\}$. Since we shall be taking combinations of longitudinal and transverse waves to solve the Riemann problem, it is convenient to define circles $C_\pm(\xi) = C_\pm(W_1(\xi))$ for $W_1(\xi) = (\xi\, e^{i\theta_0}, V_1(\xi))$, $V_1(\xi) \in \gamma_\pm$ given by (2.5) (with $\xi_1 = \xi$).

3. Solution of the Riemann Problem

To construct solutions of the Riemann problem (1.1), (1.4) under assumptions (1)–(5) of Section 1, we proceed as follows. Let W_L, W_R satisfy the *standing assumption*

$$\xi_L > 1, \qquad \xi_R > 1 \tag{3.1}$$

so that the tension is initially positive. Consider all possible combinations of centered longitudinal and transverse waves traveling left, with W_L on the left, i.e., ahead of the combination of waves. This gives states $W_-(\xi_1, \theta_1)$ on the right parameterized by only ξ_1, θ_1 and W_L. Similarly, W_R is joined to states $W_+(\xi_2, \theta_2)$ by a combination of centered waves moving right, with W_+ on the left, behind the waves and W_R on the right, ahead of the waves. The Riemann problem involves solving the four equations

$$W_-(\xi_1, \theta_1) = W_+(\xi_2, \theta_2) \tag{3.2}$$

for the unknowns ξ_k, θ_k $(k = 1, 2)$. But then $W_+ = W_-$ is the intermediate state between the combinations of waves moving left and right, and $(\xi_1, \theta_1) = (\xi_2, \theta_2)$ are the local elongation, and angle to the horizontal, of the string in this intermediate section. Consequently, system (3.2) is reduced to two equations, equating the velocity components $V_\pm$ of $W_\pm$: $V_+(\xi, \theta) = V_-(\xi, \theta)$. The intermediate angle $\theta = \theta_1 = \theta_2$ is also eliminated from these equations, as explained below. The solution of the Riemann problem thus depends upon solving a single equation for a single unknown ξ, the intermediate local elongation. This is Eq. (1.7):

$$g(\xi; W_L, W_R) = 0. \tag{3.3}$$

It remains only to define g and establish properties (i)–(iii) following (1.7). Theorem 1 will then be proved.

It is convenient to define $W_\pm(\xi, 0)$ together, with respect to a single point $W_0 = (\xi_0 e^{i\theta_0}, V_0)$, which will be replaced by W_L or W_R when defining g. Now $W_\pm(\xi, \theta)$ depends on ξ_0 in a way that can be split into three cases

(A) $\xi_0 < \xi_m$,

(B) $\xi_m < \xi_0 < \xi_M$,

(C) $\xi_M < \xi_0$.

(A) In case (A), we may join W_0 to $W_\pm(\xi, \theta)$ in such a way that longitudinal waves move faster than the (single) transverse wave, unless $\xi_m < \xi < \xi_m^*$. In the latter situation, the contact discontinuity is embedded in a rarefaction wave precisely at $\xi = \xi_m$, $x = \pm\sqrt{T'(\xi_m)}\, t$, since the

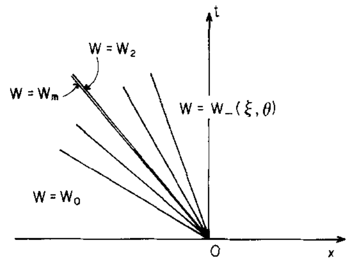

FIG. 2. A contact discontinuity embedded in a rarefaction wave, $\xi_0 < \xi_m < \xi < \xi_m^*$.

longitudinal and transverse wave speeds are the same when $\xi = \xi_m$. Such a combination of waves is shown in Fig. 2.

Now the pattern of waves joining W_0 to $W_\pm(\xi, \theta)$ can be described in the V-plane, using Fig. 3 for case (A), and Figs. 4 and 5 for cases (B), (C). Referring to Fig. 3, we have three regions separated by $C_\pm(\xi_m)$ and $C_\pm(\xi_m^*)$. Region I corresponds to values of $\xi \leqslant \xi_m$. Here, V_0 is joined to $V_\pm(\xi, \theta)$ by a longitudinal wave (specifying $V_1 = V_1(\xi) \in \gamma_\pm$), followed by a slower contact discontinuity joining $V_1(\xi)$ to $V_\pm(\xi, \theta) \in C_\pm(\xi)$. In region II, V_0 is joined to $V_\pm(\xi, \theta)$ by longitudinal waves with an embedded contact discontinuity. That is, V_0 is joined to $V_m = V_1(\xi_m) \in \gamma_\pm(W_0)$, V_m is joined to $V_2 \in C_\pm(\xi_m)$, by specifying θ and hence $W_2 = (\xi_m e^{i\theta}, V_2)$, and finally V_2 is joined to $V_\pm(\xi, \theta) \in \gamma_\pm(W_2)$. The situation in region III is similar to that in region I. V_0 is joined to $V_\pm(\xi, \theta) \in C_\pm(\xi)$ by first joining V_0 to $V_1 = V_1(\xi) \in \gamma_\pm$ by longitudinal waves, followed by a slower contact discontinuity joining V_1 to $V_\pm(\xi, \theta)$.

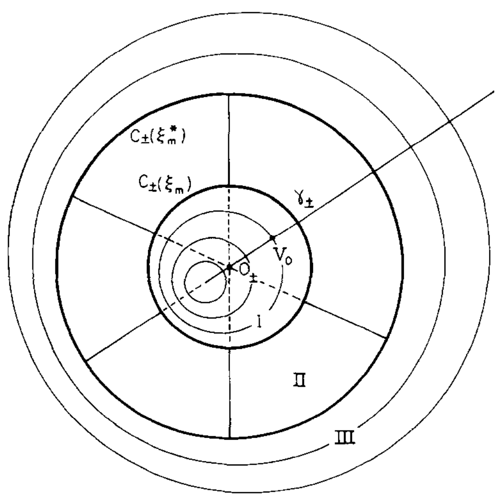

FIG. 3. The velocity plane in case (A), $\xi_0 \leqslant \xi_m$, $0_\pm = 0_\pm(\xi_m)$.

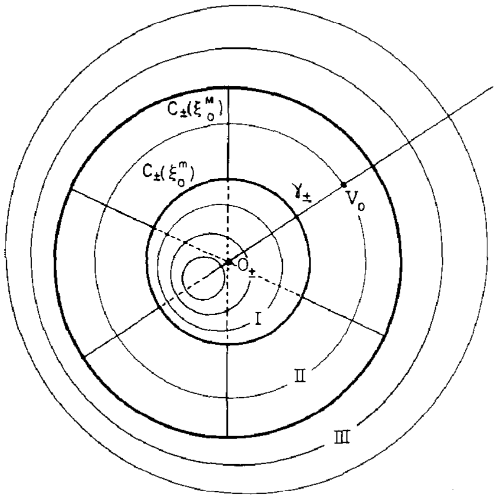

FIG. 4. The velocity plane in case (B), $\xi_m \leqslant \xi_0 \leqslant \xi_M$, $0_\pm = 0_\pm(\xi_0^m)$.

(B) Here, $\xi_m \leqslant \xi_0 \leqslant \xi_M$, and we refer to Fig. 4. The three regions I, II, III are defined by the values ξ_0^m, ξ_0^M (given by (1.9), with $\xi = \xi_0$). In regions I, III, V_0 is joined to $V_\pm(\xi, \theta)$ by longitudinal waves followed by a slower contact discontinuity. In region II, we have a new construction. Since $\xi_0^m < \xi < \xi_0^M$, V_0 is joined to $V_\pm(\xi, \theta)$ by a contact discontinuity *followed by a slower* combination of longitudinal waves.

(C) If $\xi_0 \geqslant \xi_M$, the three regions are give in Fig. 5, and are separated by

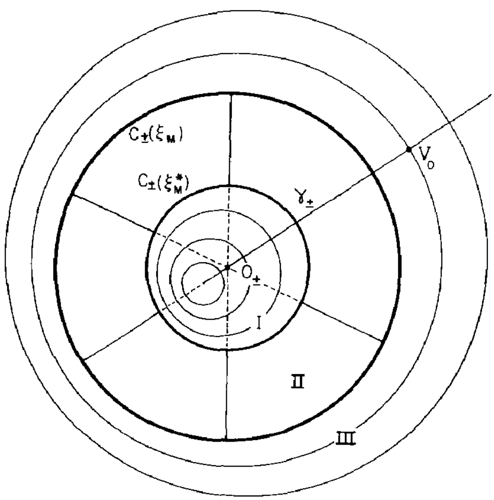

FIG. 5. The velocity plane in case (C), $\xi_0 \geqslant \xi_M$, $0_\pm = 0_\pm(\xi_M)$.

$C_\pm(\xi_M)$ and $C_\pm(\xi_M^*)$. The similarity between Figs. 3 and 5 corresponds to the fact that the way V_0 is joined to $V_\pm(\xi, \theta)$ is precisely as in case A except that ξ_m, ξ_m^* should be replaced by ξ_M^*, ξ_M respectively.

Figures (3)–(5) suggest the use of polar coordinates, and indeed the diagrams represent the image of the polar coordinate system in the $r_x = \xi\, e^{i\theta}$ plane under the mapping $V = V_\pm(\xi, \theta)$. The circles $\xi = \text{constant}$ in the r_x plane corresponding to regions I, III are mapped into circles $C_\pm(\xi)$ in the V plane, but with centers $0_\pm(\xi)$ depending on ξ. Specifically, in case (A), $1 \leqslant \xi_0 \leqslant \xi_m$, we define

$$0_\pm(\xi) = \begin{cases} V_0 \pm (B(\xi) - A(\xi, \xi_0))\, e^{i\theta_0} & \text{if} \quad \xi \leqslant \xi_m \text{ or } \xi \geqslant \xi_m^* \\ V_0 \pm (B(\xi_m) - A(\xi_m, \xi_0))\, e^{i\theta_0} & \text{if} \quad \xi_m \leqslant \xi \leqslant \xi_m^*. \end{cases}$$

Note also that $0_\pm(\xi_m) = 0_\pm(\xi_m^*)$ since $(T(\xi_m^*) - T(\xi_m))/(\xi_m^* - \xi_m) = T(\xi_m)/\xi_m = T(\xi_m^*)/\xi_m^*$.

In case (B), $\xi_m \leqslant \xi_0 \leqslant \xi_M$,

$$0_\pm(\xi) = \begin{cases} V_0 \pm (B(\xi) - A(\xi, \xi_0))\, e^{i\theta_0} & \text{if} \quad \xi \leqslant \xi_0^m \text{ or } \xi \geqslant \xi_0^M \\ V_0 \pm B(\xi_0)\, e^{i\theta_0} & \text{if} \quad \xi_0^m \leqslant \xi \leqslant \xi_0^M. \end{cases}$$

Note that $B(\xi_0^m) - A(\xi_0^m, \xi_0) = B(\xi_0^M) - A(\xi_0^M, \xi_0)$.

In case (C), $\xi_0 \geqslant \xi_M$,

$$0_\pm(\xi) = \begin{cases} V_0 \pm (B(\xi) - A(\xi, \xi_0))\, e^{i\theta_0} & \text{if} \quad \xi \leqslant \xi_M^* \text{ or } \xi \geqslant \xi_M \\ V_0 \pm B(\xi_M)\, e^{i\theta_0} & \text{if} \quad \xi_M^* \leqslant \xi \leqslant \xi_M. \end{cases}$$

Note, similarly to case (A), that $0_\pm(\xi_M^*) = 0_\pm(\xi_M)$.

As explained above, $0_\pm(\xi)$ is the center of the circle $C_\pm(\xi)$, in regions I, III. In region II, $0_\pm(\xi)$ is the point on which the radial lines are centered, corresponding to the construction in region II, in which ξ is varied after θ is varied, in going from V_0 to $V_\pm$.

In regions I, III, the circles $C_\pm(\xi)$ are nested, as shown in [5], so that $V_\pm$ is 1–1 and onto these regions, away from $\xi = 1$. In region II, $V_\pm(\xi, \theta)$ is obviously 1–1 and onto since it is 1–1 onto each radial line. Now we can give formulae for $V_\pm(\xi, \theta)$:

$$V_\pm(\xi, \theta) = 0_\pm(\xi) \pm h_\pm(\xi)\, e^{i\theta} \tag{3.5}$$

where $h_\pm(\xi) \geqslant 0$ is defined below:

Case A. $\xi_0 \leqslant \xi_m$:

$$h_\pm(\xi) = \begin{cases} B(\xi) & \text{if} \quad \xi \leqslant \xi_m \text{ or } \xi \geqslant \xi_m^* \\ B(\xi_m) + A(\xi, \xi_m) & \text{if} \quad \xi_m \leqslant \xi \leqslant \xi_m^*. \end{cases}$$

Case B. $\xi_m \leqslant \xi_0 \leqslant \xi_M$:

$$h_\pm(\xi) = \begin{cases} B(\xi) & \text{if } \xi \leqslant \xi_0^m \text{ or } \xi \geqslant \xi_0^M \\ B(\xi_0) + A(\xi, \xi_0) & \text{if } \xi_0^m \leqslant \xi \leqslant \xi_0^M. \end{cases}$$

Case C. $\xi_M \leqslant \xi_0$:

$$h_\pm(\xi) = \begin{cases} B(\xi) & \text{if } \xi \leqslant \xi_M^* \text{ or } \xi \geqslant \xi_M \\ B(\xi_M) + A(\xi, \xi_M) & \text{if } \xi_M^* \leqslant \xi \leqslant \xi_M. \end{cases}$$

Note that $h_\pm(\xi)$ depends continuously on ξ and ξ_0.

Next, define $V_-(\xi, \theta)$ with respect to $W_0 = W_L$, and define $V_+(\xi, \theta)$ with respect to $W_0 = W_R$. Then to solve the Riemann problem (1.1), (1.4), we have to solve $V_-(\xi, \theta) = V_+(\xi, \theta)$. That is, from (3.5),

$$0_+(\xi) - 0_-(\xi) = (h_+(\xi) + h_-(\xi))\, e^{i\theta}, \tag{3.6}$$

where of course $0_-, h_-$ are defined with respect of to W_L, while $0_+, h_+$ depend on W_R. Equation (3.6) specifies θ as the angle to the horizontal of the straight line between $0_-(\xi)$ and $0_+(\xi)$. In particular, the intermediate velocity $V_\pm(\xi, \theta)$ lies on this line. Taking norms of both sides of (3.6), we get a single equation for ξ:

$$g(\xi) \equiv |0_+(\xi) - 0_-(\xi)| - (h_+(\xi) + h_-(\xi)) = 0. \tag{3.7}$$

Theorem 1 will now follow from

PROPOSITION 3.1. *Under assumptions* (1)–(5) *of Section* 1, *g satisfies*

(i) $g(1) \geqslant 0$,

(ii) $\lim_{\xi \to \infty} g(\xi) < 0$,

(iii) $\partial g/\partial \xi\,(\xi) < 0$ *for all* $\xi > 1$.

Proof. Property (i) is immediate from (3.7) since $h_\pm(1) = 0$. To prove property (ii), we argue geometrically as follows, allowing for the possibility that $\xi_m^* = \infty$. Suppose first that $\xi_m^* < \infty$. Then region III is defined for both W_L, W_R. Moreover, as shown in [5], the circles $C_\pm(\xi)$, $\xi > \xi_m^*$ are nested and expand to infinity filling out the region outside $C_\pm(\xi_m^*)$. In particular, for some $\xi > \xi_m^*$, $C_+(\xi)$ and $C_-(\xi)$ must intersect nontangentially (note that they have the same radius so one cannot lie entirely within the other). For such a ξ, $g(\xi) < 0$.

Next, suppose $\xi_m^* = \infty$ and region III does not exist for W_L, but region III is defined for W_R. Then W_R must be in case B with $\xi_R^M < \infty$, or case C, whereas W_L must be in case A, or case B with $\xi_L^M = \infty$. So region II for W_L lies outside $C = C_-(\xi_m)$ in case A, or $C = C_-(\xi_L^m)$ in case B. Since the

$C_+(\xi)$ ($\xi > \xi_R^M$ in case B or $\xi > \xi^M$ in case C) are nested and fill out the V-plane except for the interior of a circle, choose $\xi > 0$ such that $C_+(\xi)$ completely encloses C. Then $g(\xi) < 0$.

Finally, if region III is defined for neither W_L nor W_R, property (ii) follows from (3.7) because $h_\pm(\xi) \to \infty$ as $\xi \to \infty$, while $0_+(\xi)$ and $0_-(\xi)$ are fixed for large enough ξ.

Proving property (iii) involves calculating $\partial g/\partial \xi$ directly. However, the formula for $g(\xi) = g(\xi; W_L, W_R)$ depends upon which of cases (A), (B), (C) W_L and W_R fall under, and to which region I, II, or III ξ corresponds. Thus, if W_L, W_R fall into the same case, there are three calculations, whereas if they fall under different cases, there are five calculations, because the formula for $g(\xi)$ then changes at four values of ξ. Rather than perform all of these 39 calculations (which may be immediately reduced to 24 due to the symmetry between left and right), just one representative pair W_L, W_R will be considered here, chosen so as to give all the analytical details required to verify property (iii) for any W_L, W_R.

LEMMA 3.1. *If $1 < \xi_L \leqslant \xi_m$ and $\xi_m \leqslant \xi_R \leqslant \xi_M$, then $\partial g(\xi)/\partial \xi < 0$ for all $\xi > 1$.*

Proof. Since W_L is in case (A), W_R in case (B), the definition of $g(\xi)$ changes at the four points $\xi_R^m \leqslant \xi_m \leqslant \xi_R^M \leqslant \xi_m^*$. That g is continuous at these points follows from the continuous dependence of $0_\pm(\xi)$, $h_\pm(\xi)$ on ξ, remarked upon above. Let $\eta(\xi) = |0_+(\xi) - 0_-(\xi)|$.

(a) $\xi \leqslant \xi_R^m$: $g(\xi) = \eta(\xi) - 2B(\xi)$.

$$\frac{\partial g}{\partial \xi} = \eta'(\xi) - 2B'(\xi) \tag{3.8}$$

now

$$\eta(\xi) = |V_R - V_L + [B(\xi) - A(\xi, \xi_R)]\, e^{i\theta_R} + [B(\xi) - A(\xi, \xi_L)]\, e^{i\theta_L}|,$$

so

$$\begin{aligned}\eta'(\xi) &= \eta^{-1}(0_+(\xi) - 0_-(\xi)) \cdot ([B'(\xi) - A_\xi(\xi, \xi_R)]\, e^{i\theta_R} \\ &\quad + [B'(\xi) - A_\xi(\xi, \xi_L)]\, e^{i\theta_L}) \\ &\leqslant |B'(\xi) - A_\xi(\xi, \xi_R)| + |B'(\xi) - A_\xi(\xi, \xi_L)|. \end{aligned} \tag{3.9}$$

We need the following estimates on A_ξ.

LEMMA 3.2 *(α) $0 < A_\xi(\xi, \xi_L) \leqslant B'(\xi)$ for $\xi \leqslant \xi_m$ and $\xi \geqslant \xi_m^*$ if $\xi_L \leqslant \xi_m$, (β) $0 < A_\xi(\xi, \xi_R) \leqslant B'(\xi)$ for $\xi \leqslant \xi_R^m$ and $\xi \geqslant \xi_R^M$ if $\xi_m \leqslant \xi_R \leqslant \xi_M$.*

Proof of Lemma 3.2.

$$(\alpha) \quad A(\xi, \xi_L) = \begin{cases} -[(T(\xi) - T(\xi_L))(\xi - \xi_L)]^{1/2} & \text{if} \quad \xi \leqslant \xi_L \leqslant \xi_M, \\ \int_{\xi_L}^{\xi} [T'(v)]^{1/2}\, dv & \text{if} \quad \xi_L \leqslant \xi \leqslant \xi_m, \\ [(T(\xi) - T(\xi_L))(\xi - \xi_L)]^{1/2} & \text{if} \quad \xi \geqslant \xi_m^*, \end{cases}$$

and $B(\xi) = [\xi\, T(\xi)]^{1/2}$.

Thus, for $\xi \leqslant \xi_L$ or $\xi \geqslant \xi_m^*$, $2A_\xi(\xi, \xi_L) = b^{-1}(b^2 + c^2) > 0$ and $2(A_\xi(\xi, \xi_L) - B'(\xi)) = b^{-1}(b^2 + c^2) - a^{-1}(a^2 + c^2)$, where $a^2 = T(\xi)/\xi \leqslant b^2 = [T(\xi) - T(\xi_L)]/(\xi - \xi_L) \leqslant c^2 = T'(\xi)$. That is,

$$\begin{aligned} 2(A_\xi(\xi, \xi_L) - B'(\xi)) &= (b^{-1} - a^{-1})\, c^2 + (b - a) \\ &\leqslant (b^{-1} - a^{-1})\, b^2 + b - a \\ &= -a^{-1}(a - b)^2 \leqslant 0. \end{aligned}$$

For $\xi_L \leqslant \xi \leqslant \xi_m$,

$$\begin{aligned} 2(A_\xi(\xi, \xi_L) - B'(\xi)) &= 2c - a^{-1}(a^2 + c^2) \\ &= -a^{-1}(a - c)^2 \leqslant 0. \end{aligned}$$

(β) is a little simpler. If $\xi_m \leqslant \xi_R \leqslant \xi_M$,

$$A(\xi, \xi_R) = \begin{cases} -[(T(\xi) - T(\xi_R))(\xi - \xi_R)]^{1/2} & \text{for} \quad \xi \leqslant \xi_R^m \\ [(T(\xi) - T(\xi_R))(\xi - \xi_R)]^{1/2} & \text{for} \quad \xi \geqslant \xi_R^M. \end{cases}$$

Thus, $2A_\xi(\xi, \xi^R) = d^{-1}(d^2 + c^2) > 0$ and

$$2(A_\xi(\xi, \xi_R) - B'(\xi)) = d^{-1}(d^2 + c^2) - a^{-1}(a^2 + c^2) \leqslant -a^{-1}(a - c)^2 \leqslant 0,$$

where $a^2 \leqslant d^2 = [T(\xi) - T(\xi_R)]/(\xi - \xi_R) \leqslant c^2$ for $\xi \leqslant \xi_R^m$ or $\xi \geqslant \xi_R^M$. This completes the proof of Lemma 3.2.

From (3.8), (3.9), and Lemma 3.2, we see that $\partial g/\partial \xi < 0$ for $\xi \leqslant \xi_R^m$.

(b) $\xi_R^m \leqslant \xi \leqslant \xi_m$: $g(\xi) = \eta(\xi) - A(\xi, \xi_R) - B(\xi) - B(\xi_R)$

$$\frac{\partial g}{\partial \xi} = \eta'(\xi) - A_\xi(\xi, \xi_R) - B'(\xi),$$

where

$$\eta(\xi) = |V_R - V_L + (B(\xi) - A(\xi, \xi_L))\, e^{i\theta_L} + B(\xi_R)\, e^{i\theta_R}|.$$

Thus

$$\eta'(\xi)=\eta^{-1}(0_+(\xi)-0_-(\xi))\cdot[B'(\xi)-A_\xi(\xi,\xi_L)]\,e^{i\theta_L}$$
$$\leqslant B'(\xi)-A_\xi(\xi,\xi_L)<B'(\xi) \qquad \text{from Lemma 3.2}(\alpha).$$

Therefore,

$$\frac{\partial g}{\partial \xi}<-A_\xi(\xi,\xi_R)\leqslant 0 \qquad \text{(by direct calculation)}$$

(c) $\xi_m\leqslant\xi\leqslant\xi_R^M: g(\xi)=\eta(\xi)-A(\xi,\xi_R)-A(\xi,\xi_m)-B(\xi_m)$

$$\eta(\xi)=|0_+(\xi_R)-0_-(\xi_m)|=\text{constant}.$$

Therefore

$$\frac{\partial g}{\partial \xi}=-A_\xi(\xi,\xi_R)-A_\xi(\xi,\xi_m)<0.$$

(d) $\xi_R^M\leqslant\xi\leqslant\xi_m^*: g(\xi)=\eta(\xi)-A(\xi,\xi_m)-B(\xi)-B(\xi_m)$.

with

$$\eta(\xi)=|V_R-V_L+B(\xi_m)\,e^{i\theta_L}+(B(\xi)-A(\xi,\xi_R))\,e^{i\theta_R}|.$$

Thus,

$$\eta'(\xi)=\eta^{-1}(0_+(\xi)-0_-(\xi))\cdot(B'(\xi)-A_\xi(\xi,\xi_R))\,e^{i\theta_R}$$
$$\leqslant|B'(\xi)-A_\xi(\xi,\xi_R)|<B'(\xi) \qquad \text{from Lemma 3.2}(\xi),$$

and so

$$\frac{\partial g}{\partial \xi}=\eta'(\xi)-A_\xi(\xi,\xi_m)-B'(\xi)<0.$$

(e) $\xi\geqslant\xi_m^*: g(\xi)=\eta(\xi)-2B(\xi)$.

$$\frac{\partial g}{\partial \xi}<0 \qquad \text{follows exactly as in (a).}$$

This completes the proof of Lemma 3.1.

The proof of Lemma 3.1 is repetitive, with the estimates on $\eta'(\xi)$ based entirely on Lemma 3.2 and the Cauchy–Schwarz inequality. For all other combinations of cases (A), (B), (C) for W_L, W_R a similar calculation may be carried out to prove property (iii), but the details are omitted here. This completes the proof of Proposition 3.1, and hence of Theorem 1.

Remark. In this paper, the Riemann problem has been solved for planar motion of an elastic string when the characteristic speeds are ordered through (1.3) except for ξ in an interval $[\xi_m, \xi_M]$. If (1.3) holds for all $\xi > 1$, the analysis simplifies considerably to that given geometrically in [5]. In the terminology used here, region II disappears and region I (or III) becomes the entire V plane. This greatly simplifies the formula for $g(\xi)$, which becomes, for each $\xi \geqslant 1$,

$$g(\xi) = |V_R - V_L + [B(\xi) - A(\xi, \xi_L)]\, e^{i\theta_L} + [B(\xi) - A(\xi, \xi_R)]\, e^{i\theta_R}| - 2B(\xi).$$

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